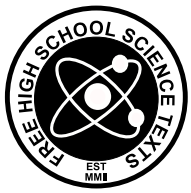


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# Contents

# Chapter 7

# Number Patterns - Grade 10

In earlier grades you saw patterns in the form of pictures and numbers. In this chapter we learn more about the mathematics of patterns.Patterns are recognisable regularities in situations such as in nature, shapes, events, sets of numbers. For example, spirals on a pineapple, snowflakes, geometric designs on quilts or tiles, the number sequence 0, 4, 8, 12, 16,....

---

**Activity :: Investigation : Patterns**
Can you spot any patterns in the following lists of numbers?

1. 2; 4; 6; 8; 10; ...
2. 1; 2; 4; 7; 11; ...
3. 1; 4; 9; 16; 25; ...
4. 5; 10; 20; 40; 80; ...

---

## 7.1 Common Number Patterns

Numbers can have interesting patterns. Here we list the most common patterns and how they are made.

Examples:

1. $1, 4, 7, 10, 13, 16, 19, 22, 25, ...$

   This sequence has a difference of 3 between each number. The pattern is continued by adding 3 to the last number each time.

2. $3, 8, 13, 18, 23, 28, 33, 38, ...$

   This sequence has a difference of 5 between each number. The pattern is continued by adding 5 to the last number each time.

3. $2, 4, 8, 16, 32, 64, 128, 256, ...$

   This sequence has a factor of 2 between each number. The pattern is continued by multiplying the last number by 2 each time.

4. $3, 9, 27, 81, 243, 729, 2187, ...$

   This sequence has a factor of 3 between each number. The pattern is continued by multiplying the last number by 3 each time.

### 7.1.1 Special Sequences

#### Triangular Numbers

$1, 3, 6, 10, 15, 21, 28, 36, 45, ...$

This sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence.

#### Square Numbers

$1, 4, 9, 16, 25, 36, 49, 64, 81, ...$

The next number is made by squaring where it is in the pattern. The second number is 2 squared ($2^2$ *or* $2 \times 2$) The seventh number is 7 squared ($7^2$ *or* $7 \times 7$) etc

#### Cube Numbers

$1, 8, 27, 64, 125, 216, 343, 512, 729, ...$

The next number is made by cubing where it is in the pattern. The second number is 2 cubed ($2^3$ *or* $2 \times 2 \times 2$) The seventh number is 7 cubed ($7^3$ *or* $7 \times 7 \times 7$) etc

#### Fibonacci Numbers

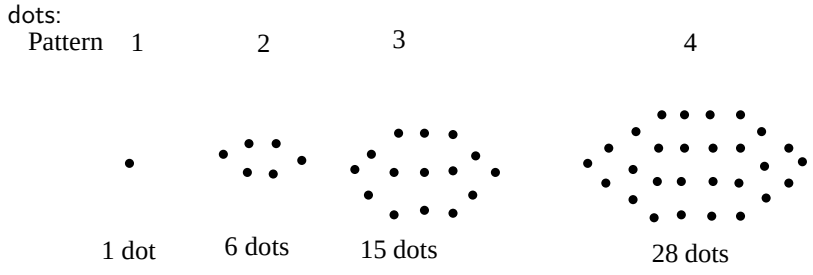
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$

The next number is found by adding the two numbers before it together. The 2 is found by adding the two numbers in front of it $(1 + 1)$ The 21 is found by adding the two numbers in front of it $(8 + 13)$ The next number in the sequence above would be 55 $(21 + 34)$

Can you figure out the next few numbers?

## 7.2 Make your own Number Patterns

You can make your own number patterns using coins or matchsticks. Here is an example using dots:



How many dots would you need for pattern 5 ? Can you make a formula that will tell you how many coins are needed for any size pattern? For example if the pattern 20? The formula may look something like

$$dots = pattern \times pattern + ...$$

**Worked Example 5: Study Table**

**Question:** Say you and 3 friends decide to study for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you and would like to sit at your table and help you study. Naturally, you move another table and add it to the existing one. Now six of you sit at the table. Another two of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you can sit comfortably.

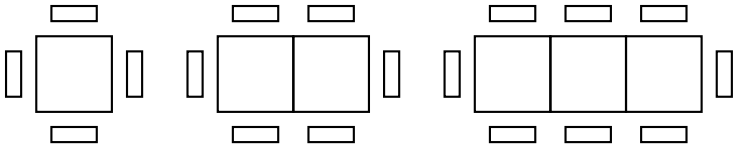


Figure 7.1: Two more people can be seated for each table added.

Examine how the number of people sitting is related to the number of tables.
**Answer**
**Step 1 : Tabulate a few terms to see if there is a pattern**

| Number of Tables, $n$ | Number of people seated |
|---|---|
| 1 | $4 = 4$ |
| 2 | $4 + 2 = 6$ |
| 3 | $4 + 2 + 2 = 8$ |
| 4 | $4 + 2 + 2 + 2 = 10$ |
| $\vdots$ | $\vdots$ |
| $n$ | $4 + 2 + 2 + 2 + \ldots + 2$ |

**Step 2 : Describe the pattern**
We can see for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two the whole time. Thus, for each table added, the number of persons increases by two.

## 7.3 Notation

A sequence does not have to follow a pattern but when it does we can often write down a formula to calculate the $n^{th}$-term, $a_n$. In the sequence

$$1; 4; 9; 16; 25; \ldots$$

where the sequence consists of the squares of integers, the formula for the $n^{th}$-term is

$$a_n = n^2 \tag{7.1}$$

You can check this by looking at:

$$\begin{aligned} a_1 &= 1^2 = 1 \\ a_2 &= 2^2 = 4 \\ a_3 &= 3^2 = 9 \\ a_4 &= 4^2 = 16 \\ a_5 &= 5^2 = 25 \\ &\dots \end{aligned}$$

Therefore, using (7.1), we can generate a pattern, namely squares of integers.

**Worked Example 6: Study Table continued ....**

**Question:** As before, you and 3 friends are studying for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you move another table and add it to the existing one. Now six of you sit at the table. Another two of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you sit comfortably as illustrated:

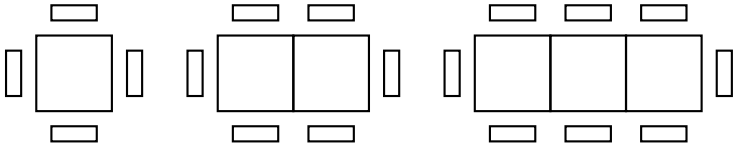


Figure 7.2: Two more people can be seated for each table added.

Find the expression for the number of people seated at $n$ tables. Then, use the general formula to determine how many people can sit around 12 tables and how many tables are needed for 20 people.

**Answer**

**Step 1 : Tabulate a few terms to see if there is a pattern**

| **Number of Tables**, $n$ | **Number of people seated** | **Formula** |
|---|---|---|
| 1 | $4 = 4$ | $= 4 + 2 \cdot (0)$ |
| 2 | $4 + 2 = 6$ | $= 4 + 2 \cdot (1)$ |
| 3 | $4 + 2 + 2 = 8$ | $= 4 + 2 \cdot (2)$ |
| 4 | $4 + 2 + 2 + 2 = 10$ | $= 4 + 2 \cdot (3)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | $4 + 2 + 2 + 2 + \ldots + 2$ | $= 4 + 2 \cdot (n - 1)$ |

**Step 2 : Describe the pattern**

The number of people seated at $n$ tables is:

$$a_n = 4 + 2 \cdot (n - 1)$$

**Step 3 : Calculate the $12^{\text{th}}$ term**

Using the general formula (36.1) and considering the example from the previous section, how many people can sit around, say, 12 tables? We are looking for $a_{12}$, that is, where $n = 12$:

$$\begin{aligned} a_n &= a_1 + d \cdot (n - 1) \\ a_{12} &= 4 + 2 \cdot (12 - 1) \\ &= 4 + 2(11) \\ &= 4 + 22 \\ &= 26 \end{aligned}$$

**Step 4 : Calculate the number of terms if $a_n = 20$**

$$\begin{aligned} a_n &= a_1 + d \cdot (n-1) \\ 20 &= 4 + 2 \cdot (n-1) \\ 20 - 4 &= 2 \cdot (n-1) \\ 16 \div 2 &= n - 1 \\ 8 + 1 &= n \\ n &= 9 \end{aligned}$$

**Step 5 : Final Answer**
26 people can be seated at 12 tables and 9 tables are needed to seat 20 people.

It is also important to note the difference between $n$ and $a_n$. $n$ can be compared to a place holder, while $a_n$ is the value at the place "held" by $n$. Like our "Study Table"-example above, the first table (Table 1) holds 4 people. Thus, at place $n = 1$, the value of $a_1 = 4$, and so on:

| $n$ | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| $a_n$ | 4 | 6 | 8 | 10 | ... |

**Activity :: Investigation : General Formula**

1. Find the general formula for the following sequences and then find $a_{10}$, $a_{50}$ and $a_{100}$:
   (a) $2, 5, 8, 11, 14, \ldots$
   (b) $0, 4, 8, 12, 16, \ldots$
   (c) $2, -1, -4, -7, -10, \ldots$
2. The general term has been given for each sequence below. Work out the missing terms.
   (a) $0; 3; ...; 15; 24$ $\quad n^2 - 1$
   (b) $3; 2; 1; 0; ...; 2$ $\quad -n + 4$
   (c) $11; ...; 7; ...; 3$ $\quad -13 + 2n$

## 7.3.1 Patterns and Conjecture

In mathematics, a conjecture is a mathematical statement which appears to be true, but has not been formally proven to be true under the rules of mathematics. Other words that have a similar in meaning to conjecture are: hypothesis, theory, assumption and premise.
For example: Make a **conjecture** about the next number based on the pattern $2; 6; 11; 17 : ...$
The numbers increase by 4, 5, and 6.
**Conjecture:** The next number will increase by 7. So, it will be $17 + 7$ or 24.

**Worked Example 7: Number patterns**

**Question:** Consider the following pattern.

$$\begin{aligned} 1^2 + 1 &= 2^2 - 2 \\ 2^2 + 2 &= 3^2 - 3 \\ 3^2 + 3 &= 4^2 - 4 \\ 4^2 + 4 &= 5^2 - 5 \end{aligned}$$

1. Add another two rows to the end of the pattern.
2. Make a conjecture about this pattern. Write your conjecture in words.
3. Generalise your conjecture for this pattern (in other words, write your conjecture algebraically).
4. Prove that your conjecture is true.

**Answer**
**Step 1 : The next two rows**

$$\begin{aligned} 5^2 + 5 &= 6^2 - 6 \\ 6^2 + 6 &= 7^2 - 7 \end{aligned}$$

**Step 2 : Conjecture**
Squaring a number and adding the same number gives the same result as squaring the next number and subtracting that number.
**Step 3 : Generalise**
We have chosen to use $x$ here. You could choose any letter to generalise the pattern.

$$x^2 + x = (x+1)^2 - (x+1)$$

**Step 4 : Proof**

$$Left\ side: \ x^2 + x$$

$$Right\ side: \ (x+1)^2 - (x+1)$$

$$\begin{aligned} Right\ side &= x^2 + 2x + 1 - x - 1 \\ &= x^2 + x \\ &= left\ side \end{aligned}$$

$$Therefore\ x^2 + x = (x+1)^2 - (x+1)$$

## 7.4 Exercises

1. Find the $n^{\text{th}}$ term for: $3, 7, 11, 15, \ldots$
2. Find the general term of the following sequences:

(a) $-2,1,4,7,\ldots$
(b) $11, 15, 19, 23, \ldots$
(c) $x-1,2x+5,5x+1,\ldots$
(d) sequence with $a_3 = 7$ and $a_8 = 15$
(e) sequence with $a_4 = -8$ and $a_{10} = 10$

3. The seating in a section of a sports stadium can be arranged so the first row has 15 seats, the second row has 19 seats, the third row has 23 seats and so on. Calculate how many seats are in the row 25.

4. Consider the following pattern:

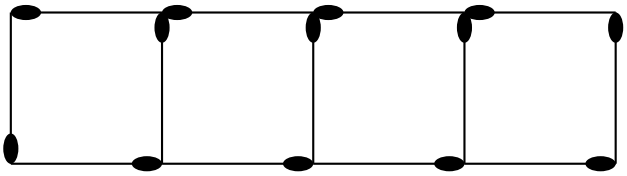
$$\begin{aligned} 2^2 + 2 &= 3^2 - 3 \\ 3^2 + 3 &= 4^2 - 4 \\ 4^2 + 4 &= 5^2 - 5 \end{aligned}$$

(a) Add at least two more rows to the pattern and check whether or not the pattern continues to work.
(b) Describe in words any patterns that you have noticed.
(c) Try to generalise a rule using algebra i.e. find the general term for the pattern.
(d) Prove or disprove that this rule works for all values.

5. The profits of a small company for the last four years has been: R10 000, R15 000, R19 000 and R23 000. If the pattern continues, what is the expected profit in the 10 years (i.e. in the $14^{\text{th}}$ year of the company being in business)?

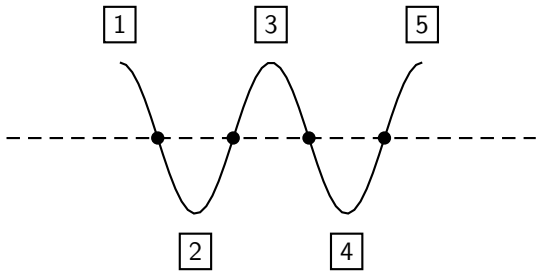
6. A single square is made from 4 matchsticks. Two squares in a row needs 7 matchsticks and 3 squares in a row needs 10 matchsticks. Determine:

(a) the first term
(b) the common difference
(c) the formula for the general term
(d) how many matchsticks are in a row of 25 squares



7. You would like to start saving some money, but because you have never tried to save money before, you have decided to start slowly. At the end of the first week you deposit R5 into your bank account. Then at the end of the second week you deposit R10 into your bank account. At the end of the third week you deposit R15. After how many weeks, do you deposit R50 into your bank account?

8. A horizontal line intersects a piece of string at four points and divides it into five parts, as shown below.



If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at four points, find the number of parts into which the string will be divided.

# Appendix A

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